B.Sc. Part III (Hons) 5th Paper

Infinite Products (contd.)

Formula

* The infinite product $\prod_{1}^{\infty} (1+U_n)$ is said to be absolutely convergent if $\prod_{1}^{\infty} (1+|U_n|)$ is convergent, U_n may be either positive or negative.

* The necessary and sufficient condition for the absolute convergence of the infinite product $\prod_{1}^{\infty} (1+U_n)$ is that $\sum |U_n|$ should be convergent.

* $\prod_{1}^{\infty} (1+|U_n|)$ converges $\Rightarrow \prod(1+U_n)$ is also convergent.

Sums

1. Prove that the infinite product $\left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{2^2}\right)\cdots\left(1+\frac{1}{n^2}\right)\ldots$ is absolutely convergent.

Soln. The given infinite product
$= \left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{2^2}\right)\cdots\left(1+\frac{1}{n^2}\right) = \prod_{1}^{\infty}\left(1+\frac{1}{n^2}\right)$

Here, $U_n = \frac{1}{n^2}$

$\Rightarrow \sum U_n = \sum \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots$

$\therefore \sum U_n$ is convergent because $\sum \frac{1}{n^p}$ is cgt if $p>1$

$\Rightarrow$ Also, $|U_n| = \left|\frac{1}{n^2}\right|$

$\Rightarrow \sum |U_n| = \frac{1}{1^2} + \frac{1}{2^2} + \cdots$ which is also convergent.

$\Rightarrow \prod_{1}^{\infty} (1+U_n)$ is absolutely convergent.

Q. Prove that $\prod_{2}^{\infty} \left\{1 + \frac{(-1)^n}{n^\alpha}\right\}$ is convergent if $\alpha > \frac{1}{2}$.

Soln Here, $U_n = \frac{(-1)^n}{n^\alpha} \Rightarrow U_n^2 = \frac{(-1)^{2n}}{n^{2\alpha}} = \frac{1}{n^{2\alpha}}$

$\Rightarrow \sum U_n^2 = \frac{1}{1^{2\alpha}} + \frac{1}{2^{2\alpha}} + \frac{1}{3^{2\alpha}} + \cdots$

which is convergent if $2\alpha > 1$ i.e. $\alpha > \frac{1}{2}$.

Now, $\sum U_n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n^\alpha} = \frac{1}{2^\alpha} - \frac{1}{3^\alpha} - \frac{1}{4^\alpha} + \cdots$

which is convergent for all values of α.

$\Rightarrow \sum U_n$ and $\sum U_n^2$ both are convergent if $\alpha > \frac{1}{2}$.

Hence $\prod_{2}^{\infty} \left\{1 + \frac{(-1)^n}{n^\alpha}\right\}$ is convergent if $\alpha > \frac{1}{2}$.